ON THE MINIMAL BASIS OF A COMPLETELY SEPARATING MATRIX(1)

BY

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ABSTRACT

We prove that there is essentially only one $p \times 2^s$ incidence matrix satisfying a given separation condition, which is of minimal degree, and we find that matrix.

Introduction. A matrix (m_i^j) whose entries are either 0 or 1 is called *completely separating* if for every pair of columns j_1, j_2 there are two rows i_1, i_2 such that $m_{i_1}^{j_1} = m_{i_2}^{j_2} = 0$ and $m_{i_1}^{j_2} = m_{i_2}^{j_1} = 1$.

In [2] M. Maschler and B. Peleg proved that the degree of such a matrix of n columns is $\geqq \lambda_n$ where $\lambda_n = 2 + [\log_2(n - 1/2)]$ and gave an example of a completely separating matrix of n columns whose degree is precisely λ_n.

In this paper we offer a different proof of the above result, and in addition we show that for $n = 2^s$ the example given in [2] is the *only* matrix of minimal degree, up to addition of superfluous columns.

This combinatorial problem, in addition to its intrinsic interest, may be useful in studying the properties of those kernels which have a maximal dimension.

1. Definitions and Notation.

Definition 1.1. A matrix $M^{p \times n} = (m_i^j)$ of p rows and n columns is called a *completely separating incidence matrix* (c.s.i.m.) if $m_i^j = 0$ or 1 for all $i = 1, \cdots, p$; $i = 1, \cdots, n$, and if for every pair of *columns* m^{j_1}, m^{j_2} there exists a pair of *rows* m_{i_1}, m_{i_2} such that

$$m_{i_1}^{j_1} = m_{i_2}^{j_2} = 1 \text{ and } m_{i_1}^{j_2} = m_{i_2}^{j_1} = 0.$$

Definition 1.2. A matrix $B^{q \times n} = (b_i^j)$ is called a *completely separating incidence basis* (c.s.i.b.) if $b_i^j = 0$ or 1, $i = 1, \cdots, q$ and $j = 1, \cdots, n$ and if there exists *some* completely separating incident matrix $M^{p \times n}$ for which $B^{q \times n}$ is a basis.

Definition 1.3. $B^{q \times n}$ is called a *minimal completely separating incidence basis* if it is a c.s.i.b. and if $B^{q' \times n}$ is any other c.s.i.b., then $q' \geqq q$.

Definition 1.4. A matrix $B^{q \times n}$ will be said to satisfy a 2 *condition* if any two rows of B have at most 2^{q-2} identical components; i.e. if b_{i_1}, b_{i_2} are any 2 rows of B then the set $J = \{j : b_{i_1}^j = b_{i_2}^j\}$ has at most 2^{q-2} components.

Definition 1.5. A matrix $B^{q \times 2^s} = B$ will be said to satisfy a 0-*condition* if every row of B has at most 2^{s-1} zero components.

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NOTATION.

1.6 — We shall denote the *i*th *row* of the matrix $B^{q\times n} \equiv (b_i^j)$ by b_i and the *j*th *column* by b^j.

1.7 — We shall denote by A^s the set of all the 2^s possible column vectors of length s whose components are either zero or one. Thus $A^2 = \left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right\}$

1.8 — We shall denote by $B_1^{(s+1)\times 2^s} = (b_i^j)$ the following matrix: $b_i = (1,\cdots,1,0,\cdots,0,\cdots,1,\cdots,1,0,\cdots,0)$ for $i = 1,\cdots,s$ where we have 2^{s-i} ones followed by 2^{s-i} zeroes, repeated 2^{i-1} times, and $b_{s+1}^j = 1$ for all j. For example, if $s = 3$, $B_1^{4\times 8}$ is the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

1.9 — We shall denote by $B_2^{(s+1)\times 2^s}$ the matrix whose first s rows are identical to the first s rows of $B_1^{(s+1)\times 2^s}$ and whose last row $b_{s+1} = (0,1,0,1,\cdots,0,1,0,1)$; i.e., $b_{s+1}^{2j-1} = 0$, $b_{s+1}^{2j} = 1$, $j = 1,2,\cdots,2^{s-1}$.

1.10 — We shall write $B \sim C$ if the matrix C can be obtained from the matrix B by column and row permutations.

In the following section we prove that the degree of a minimal c.s.i.b. with $n = 2^s$ columns is $s+1$. In section 4 we determine the degree of a minimal c.s.i.b. for any n. We also prove that for $n = 2^s$ columns there is only one c.s.i.m. M of minimal degree, which is itself minimal in the sense that no proper subset of rows of M is completely separating, and we characterize that matrix completely.

2. Properties of a Completely Separating Basis.

LEMMA 2.1. *Let b^{j_1} and b^{j_2} be any two columns of a c.s.i.b. $B \equiv B^{q\times n}$. Then the vector sum $(b^{j_1}) + (b^{j_2})$ is not a column of B.*

Proof by contradiction. B is a basis of a c.s.i.m. M; let b^{j_1}, b^{j_2}, b^{j_3} be three columns of B such that $(b^{j_1}) + (b^{j_2}) = (b^{j_3})$. Then $b_i^{j_2} = 1 \Rightarrow b_i^{j_3} = 1$ for all $i = 1,\cdots,q$. Since M is completely separating and B is its basis, there exists a row vector $m_k = \sum_s c_s b_s = (m_k^1,\cdots,m_k^n) \in M$ such that $m_k^{j_2} = 1$ and $m_k^{j_3} = 0$. But $m_k^{j_1} + m_k^{j_2} = \sum_s c_s(b_s^{j_1} + b_s^{j_2}) = \sum_s c_s b_s^{j_3} = m_k^{j_3} = 0$. Therefore $m_k^{j_1} = -1$ contrary to definition of M as an incidence matrix.

LEMMA 2.2. *Let $B^{q\times n} = B$ be a c.s.i.b. Then* (1) *no two columns of B are identical and* (2) *no column of B is identically* 0.

Proof by contradiction. B is a basis of a c.s.i.m. M. (1) Suppose $(b^j) \equiv (b^k)$, $j \neq k$. Then $(m^j) \equiv (m^k)$, $j \neq k$ are two columns of M, contrary to the definition of M as completely separating. (2) Suppose $(0,\cdots,0)' \in B$. Then $(0,\cdots,0)'$ is a column of M contrary to the definition of a c.s.i.m.

LEMMA 2.3. *Let $B^{q\times 2^s} = B$ be a c.s.i.b. Then $q \geqq s+1$.*

Proof. Suppose $q < s$. Consider $A^q =$ set of $2^q < 2^s$ different possible columns of length q. Since B has 2^s columns. it must have two identical columns, contrary to lemma 2.2.

Suppose $q = s$. Then $2^q = 2^s$. Therefore either B contains two identical columns or it contains *all* the 2^s columns of A^s. But $(0,\cdots,0)' \in A^s$; therefore we have a contradiction to lemma 2.2. Thus $q \geqq s+1$.

LEMMA 2.4. $B_i^{(s+1)\times 2^s}$, $i = 1,2$ *(see notation* 1.8) *are completely separating incidence bases.*

Proof. Consider $B_1^{(s+1)\times 2^s}$. Define the matrix $M^{(2s+1)\times 2^s} = M = (m_i^j)$ by $m_i = b_i$, $i = 1,\cdots,s+1$. $m_{s+1+i} = b_{s+1} - b_i$, $i = 1,\cdots,s$. The rows of $B_1^{(s+1)\times 2^s}$ are linearly independent. Indeed, suppose there exist c_i such that $\sum_i c_i b_i = (0,\cdots,0)$. Then since $B_1^{(s+1)\times 2^s}$ contains *every possible* column vector of length $s+1$ whose last component is one, and whose other components are either zero or one, we have: $c_i + c_{s+1} = c_i + c_j + c_{s+1} = 0$ for all $j = 2,\cdots,s$. and all $i \neq j$, $s+1$. Therefore $c_j = 0$ for $j = 2,\cdots,s$, and $c_{s+1} = 0 = c_1$.

Obviously B is a basis of M.

Let m^{j_1}, m^{j_2} be any two columns of M. Since no two columns of B are identical, there exists some row $b_{i_0} = m_{i_0}$ in which $m_{i_0}^{j_1} = 1$ and $m_{i_0}^{j_2} = 0$, (or $m_{i_0}^{j_1} = 0$; and $m_{i_0}^{j_2} = 1$), $i_0 \leqq s+1$. Therefore

$$m_{s+1+i_0}^{j_1} = 1 - m_{i_0}^{j_1} = 0 \text{ and } m_{s+1+i_0}^{j_2} = 1 - m_{i_0}^{j_2} = 1.$$

Therefore M is completely separating.

$B_2^{(s+1)\times 2^s}$ is also a basis of the above defined M. Therefore $B_i^{(s+1)\times 2^s}$, $i = 1,2$ are completely separating bases.

LEMMA 2.5. *If $B^{q\times 2^s}$ is a minimal c.s.i.b., then $q = s+1$.*

The proof follows immediately from the two preceding lemmas.

LEMMA 2.6. *Let the incidence matrix $B^{(s+1)\times 2^s} \equiv B \equiv (b_i^j)$ be a basis of the c.s.i.m. $M^{(q\times 2^s)} = M = (m_i^j)$. Then $m = \sum_i c_i b_i \in M \Rightarrow c_i = 0, 1$ or -1 for $i = 1,\cdots,s+1$.*

Proof. Let B_i be the matrix obtained from B by deleting row b_i. Let $(b^p)_i = (b_1^p,\cdots,b_{i-1}^p, b_{i+1}^p,\cdots)'$; i.e., the pth column of B_i. We distinguish two cases:

Case (1): B_i has no two identical columns; thus all 2^s different columns of A^s belong to B_i. In particular $(b^{p_i})_i = (0\ \cdots,0)'$ for some p_i. Therefore, by

Lemma 2.2, $(0,\cdots,0,1,0,\cdots,0)' = b^{p_i}$. Let $m_k = \Sigma_s c_s b_s = (m_k^1,\cdots, m_k^{2^s}) \in M$. $m_k^{p_i} = \Sigma_s c_s b_s^{p_i} = c_i b_i^{p_i} = c_i$. Therefore, since M is an incidence matrix, c_i is equal to 0 or 1.

CASE (2). B_t has two identical columns, say $(b^{p_i})_t = (b^{q_i})_t$, $p_i \neq q_i$. By Lemma 2.2 we can assume that $b_i^{p_i} = 1$, $b_i^{q_i} = 0$. Let $m_k = (m_k^1, \cdots, m_k^{2^s}) = \Sigma_s c_s b_s \in M$. Since $m_k^j = 0$ or 1 for all $j = 1,\cdots,2^s$ we have $|m_k^{p_i} - m_k^{q_i}| = 0$ or 1. But $|m_k^{p_i} - m_k^{q_i}| = |\Sigma_s c_s(b_s^{p_i} - b_s^{q_i})| = |c_i(b_i^{p_i} - b_i^{q_i})| = |c_i| = 0$ or 1. Therefore $c_i = 0$, 1 or -1.

LEMMA 2.7. *Let $B^{(s+1)\times 2^s} \equiv B$ be any matrix containing two row vectors b_i, b_j whose sum is the unit vector. Let B_t be the matrix obtained from B by deleting the row vector b_i. If no two columns of B_t are identical, then $B \sim B_2^{(s+1)\times 2^s}$. See* 1.10).

Proof. The 2^s columns of B_t comprise the set A^s (See 1.7) because no two columns are identical. Thus appropriate row and column permutation will bring B into the form $B_2^{(s+1)\times 2^s}$.

The most difficult part of this work is

THEOREM 2.8. *$B_i^{(s+1)\times 2^s}$, $i = 1,2$, and those matrices obtained from $B_i^{(s+1)\times 2^s}$ by row and column permutations, are the only minimal c.s.i.b. having $n = 2^s$ columns.*

Since the proof of this theorem begins in this section and continues in section 4, we shall first give an outline of the proof.

We restrict ourselves to those c.s.i.b. which satisfy a 0-condition and a 2-condition; this involves no loss of generality, because we shall prove that any c.s.i.b. having 2^s columns must satisfy these conditions. Under this restriction we consider the following cases, where in each case $B^{(s+1)\times 2^s} \equiv B$ is a c.s.i.b.:

Case a. The unit row vector $(1,\cdots 1) \in B$.[2]

$B \sim B_1^{(s+1)\times 2^s}$ (Lemma 2.9)

Case b. The unit column vector $(1,\cdots 1)' \in B$.
$B \sim B_1^{(s+1)\times 2^s}$ (Lemma 2.10)

Case c. There exists a row vector $b_i \in B$ such that $\Sigma_i b_i^j = 2^{s-1}$.
$B \sim B_i^{(s+1)\times 2^s}$ for $i = 1$ or 2 (Theorem 2.8, part B)

Case d. There exists a row vector $b_i \in B$ such that $\Sigma_i b_i^j < 2^{s-1}$.
This case cannot occur. (Theorem 2.8, part A)

Case e. There exists a row vector $b_i \in B$ such that $\Sigma_i b_i^j > 2^{s-1}$.
This case cannot occur. (Theorem 2.8, part E)

(2) We shall write $b_k \in B$ if b_k is a row vector in the matrix B, and similarly $b^k \in B$ if b^k is column vector in B.

LEMMA 2.9. *If* $B^{(s+1)\times 2^s} \equiv B$ *is any c.s.i.b. containing the unit row vector* $(1,\cdots,1)$, *then* $B_1^{(s+1)\times 2^s} \sim B$.

Proof. Permute rows of B so that $(1,\cdots,1)$ is the last row. B contains 2^s different columns, all of which belong to A^{s+1}. But A^{s+1} has exactly 2^s columns whose last components is *one*. Therefore B contains exactly those 2^s columns. Similarly $B_1^{(s+1)\times 2^s}$ contains the same 2^s columns. Therefore $B_1^{(s+1)\times 2^s} \sim B$.

LEMMA 2.10. If $B^{(s+1)\times 2^s} \equiv B$ is a c.s.i.b. satisfying a 0-*condition* (Def. 1.5) and containing the unit column vector $(1,\cdots,1)'$, then $B \sim B_1^{(s+)\times 2^s}$.

Proof. Without loss of generality we can assume that $b^1 = (1,\cdots,1)' \in B$. Then $b^k \notin B \Rightarrow b^1 - b^k \in B$ (Lemma 2.1). But A^{s+1} is composed of 2^s pairs of columns of the type b^k, $b^1 - b^k$. Thus B contains exactly one of each pair; i.e., $b^k \in B \Leftrightarrow b^1 - b^k \notin B$ for all $b^k \in A^{s+1}$. B is a basis of some completely separating M. Therefore, for any fixed j, $j \neq 1$, there exists an $m_t = (m_t^1, \cdots, m_t^{2^s}) = \sum_i c_i b_i \in M$ such that $m_t^1 = 0 = \sum_{i=1}^{s+1} c_i$ and $m_t^j = 1$ for some $j \neq 1$. We shall establish three propositions:

PROPOSITION I. There exist i_0, j_0 such that $c_{i_0} = 1$, $c_{j_0} = -1$, $c_i = 0$ for all $i \neq i_0, j_0$.

Indeed, there exist i_0, j_0 such that $c_{i_0} = 1$, $c_{j_0} = -1$, because $\sum_i c_i = 0$, some $c_i \neq 0$, and all $c_i = 0$, 1 or -1 (Lemma 2.7). Suppose proposition I is not true. Then there exists $c_{i_1} = 1$, $i_1 \neq i_0$ (or there exists $c_{j_1} = -1$, $j \neq j_0$). For simplicity permute rows so that $i_0 = 1$, $j_0 = 2$, $i_1 = 3$.

Let $b^k = (1,0,1,0,0,\cdots,0,0)$.

Then $b^1 - b^k = (0,1,0,1,1,\cdots,1,1)$.

$$b^k \in B \Rightarrow m_t^k = c_1 + c_3 = 2.$$

$$b^1 - b^k \in B \Rightarrow m_t^k = \sum_{i \neq 1,3} c_i = \sum_i c_i - c_1 - c_3 = 0 - 2 = -2.$$

But either b^k or $b^1 - b^k$ must belong to B, and in either case we get a contradiction since M is an incidence matrix. A similar argument shows there does not exist $c_{j_1} = -1$, $j_1 \neq j_0$.

PROPOSITION II. B contains 2^{s-1} columns b^k such that $b_{i_0}^k = 1$ and $b_{j_0}^k = 0$, where i_0, j_0 are as defined above.

Let a^k be any of the 2^{-1} column vectors in A^{s-1}, such that $a_{j_0}^k = 1$ and $a_{i_0}^k = 0$. Then $a^k \notin B$ (because $a^k \in B \Rightarrow m_t^k = c_{j_0} = -1$). Therefore *all of the* 2^{s-1} columns of form $1 - a^k = b^k$, where $b_{j_0}^k = 0$ and $b_{i_0}^k = 1$, belong to B.

PROPOSITION III. $b_{i_0} = (1,\cdots,1) \in B$.

By Proposition II, b_{j_0} has at least 2^{s-1} zeroes. By the 0-condition satisfied by B, b_{j_0} has at most 2^{s-1} zeroes. Therefore b_{j_0} has exactly 2_s^{s-1} zeroes. I.e., there exist exactly 2^{s-1} columns in B such that $b^k_{j_0} = 1$, and therefore in these columns $b^k_{i_0} = 1$; there also exist 2^{s-1} columns b^k such that $b^k_{j_0} = 0$ and $b^k_{i_0} = 1$. I.e., $b_{i_0} = (1, \cdots, 1)$.

Proof of Lemma 2.10. Since $(1, \cdots, 1) \in B$, $B \sim B_1^{(s+1)\times 2^s}$. (Lemma 2.9).

Lemma 2.11. If $B^{(s+1)\times 2^s} \equiv B$ is a c.s.i.b. satisfying the 2 condition and the 0-condition (Def. 1.4–5) and if $(1, \cdots, 1) \notin B$ and if B contains some column vector which is everywhere zero except in one component, then $B \sim B_2^{(s+1)\times 2^s}$.

Proof. Without loss of generality, let $b^1 = (1, 0, 0, \cdots, 0)' \in B$. Let a^k be any column of A^{s+1} for which $a^k_1 = 1$. Let $a^{k*} = a^k - b^1$. Then, by Lemma 2.1, $a^k \in B \Leftrightarrow a^{k*} \notin B$.

Let $B_{\hat{1}}$ be the matrix obtained from B by deleting row 1 of B. Then since $B_{\hat{1}}$ has no two identical columns, it contains all 2^s columns of A^s, and in particular $(1, \cdots, 1)' \in B_{\hat{1}}$; therefore $b^{j_0} = (0, 1, 1, \cdots, 1)' \in B$, for some j_0, since $(1, \cdots, 1) \notin B \Rightarrow (1, \cdots, 1)' \notin B$. (Lemma 2.10).

We will prove $b_1 = (1, \cdots, 1) - b_k$, which will complete the proof, applying Lemma 2.7. We distinguish two cases.

Case (1). There exists a column $b^p \neq b^{j_0} = (0, 1, \cdots, 1)'$, $b^p \in B$ such that $b^p_1 = 0$.

Case (2). $b^p_1 = 1$, all $p \neq j_0$.

Case (1). Since B is the basis of a completely separating M, there exists $m_k = (m_k^1, \cdots, m_k^{2^s}) = \sum c_i b_i \in M$ such that $m_k^p = 1$ and $m_k^{j_0} = 0$. We shall prove three propositions:

Proposition. I. $c_1 = 1$.

From the beginning of the proof of this lemma, we know that there exists $(b^q)_{\hat{1}} \in B_{\hat{1}}$ such that $b^p_i + b_i^q = 1$, all $i \neq 1$. But $m^{j_0}_t = \sum c_i b^{j_0}_i = \sum_2^{s+1} c_i = 0$, and $m^p_t = \sum_1^{s+1} c_i b^p_i = \sum_2^{s+1} c_i b^p_i = 1$. Therefore $\sum_2^{s+1} c_i b^q_i = -1$.

But $m^q_t = \sum_1^{+1} c_i b^q_i = -1 + c_i b^q_1 = 0$ or 1. Therefore $c_1 b^q_1 = 1$ or 2; i.e., $b^q_1 = 1$ and $c_1 = 1$ or 2. But $(1, 0, \cdots, 0)' \in B \Rightarrow m^1_t = c_1 = 0$ or 1. Therefore $c_1 = 1$.

Proposition II. There exists exactly *one* c_{i_0} which is equal to -1.

Obviously there exists at least one $c_{i_0} = -1$. Suppose $c_{i_0} = c_{i_1} = -1$, $i_0 \neq i_1$. Since $B_{\hat{1}}$ contains all the columns of A^s, there exists $(b^k)_{\hat{1}} \in B$ for which $b^k_{i_0} = b^k_{i_1} = 1$ and $b^k_i = 0$, all $i \neq i_0, i_1, 1$.

$$m^k_t = c_1 b^k_1 + c_{i_0} b^k_{i_0} + c_{i_1} b^k_{i_1} = b^k_1 - b^k_{i_0} - b^k_{i_1} = b^k_1 - 2 < 0,$$

because $b^k_1 = 0$ or 1. But $m^k_t = 0$ or 1 and we have a contradiction.

Proposition III. $b_1 = (1, \cdots, 1) - b_{k_0}$ for some k_0, $1 < k_0 \leqq s + 1$. Indeed, $\sum_{i=2}^{s+1} c_i = 0$ and $c_{i_0} = -1$, $2 \leqq i_0 \leqq s + 1$ and $c_i = 0$ or 1 for all i, $2 \leqq i \leqq s + 1$ and $i \neq i_0$, by Lemma 2.6 and Propositions I–II of this lemma. Therefore there exists a k_0 such that $c_{k_0} = 1$, $2 \leqq k_0 \leqq s+1$, and $c_i = 0$, $i \neq 1$, i_0, k_0. Since $B_{\hat{1}}$ contains all the 2^s columns of A^s we know that there exist:

a) 2^{s-2} columns $b^k \in B$ such that $b^k_{i_0} = 1$ and $b^k_{k_0} = 0$

b) " " " " $b^k_{i_0} = 0$ $b^k_{k_0} = 1$

c) " " " " $b^k_{i_0} = 1$ and $b^k_{k_0} = 1$

d) " " " " $b^k_{i_0} = 0$ $b^k_{k_0} = 0$.

In case (a) $m^k_t = c_1 b^k_1 + c_{i_0} b^k_{i_0} = b^k_1 - 1 = 0$ or $1 \Rightarrow b^k_1 = 1$.

(b) $m^k_t = c_1 b^k_1 + c_{k_0} b^k_{k_0} = b^k_1 + 1 = 0$ or $1 \Rightarrow b^k_1 = 0$.

Therefore rows b_{i_0} and b_1 have at *least* 2^{s-1} identical components, and since B satisfies a 2-condition they have at *most* 2^{s-1} identical components. Therefore in case (c) $b^k_{i_0} = 1 \neq b^k_1 = 0$

(d) $b^k_{i_0} = 0 \neq b^k_1 = 1$.

Therefore $b_{k_0} + b_1 = (1, \cdots, 1)$.

Case (2). Consider some column (b^p). There exists $(b^q)_{\hat{1}} \in B_{\hat{1}}$ such that $(b^p)_{\hat{1}} + (b^q)_{\hat{1}} = (1, \cdots, 1)'$. There exists $m_k = \sum c_i b_i = (m^1_k, \cdots, m^{2^s}_k) \in M$ such that $m^1_k = 1$ and

$$m^p_k = 0. \; m^1_k = \sum c_i b^1_i = c_1 b_1 = c_1 = 1.$$

$$m^p_k = 0 = \sum_2^{s+1} c_i b^p_i + c_1 b^p_1 = \sum_2^{s+1} c_i b^p_i + 1 \Rightarrow \sum_2^{s+1} c_i b^p_i = -1.$$

$$\sum_2^{s+1} c_i (b_i^p + b^q_i) = \sum_2^{s+1} c_i = m^{j_0}_k = 0 \text{ or } 1.$$

Therefore $\sum_2^{s+1} c_i b^q_i = 1$ or 2, and $\sum_1^{s+1} c_i b_i^q = c_1 b_1^q + \sum_2^{s+1} c_i b^q_i = m^q_k = 2$ or 3. Thus we get a contradiction since M is an incidence matrix, and Case (2) cannot occur.

3. **Inequalities.** We now list a few inequalities involving integers which are needed to complete the proof of theorem 2.8. The proofs are simple and will be omitted.

LEMMA 3.1. If $0 < \binom{2p+k}{p+r} < 2^{2p+s}$ for some $p = p_0$, where p, k and r are integers, and if $(k-2r)^2 \leqq 2p + k + 2$, then $\binom{2p+k}{p+r} < 2^{2p+s}$ for all $p \gtreqless p_0$.

Proof. By induction on p.

LEMMA 3.2. *If q, k are integers and $a = 0$ or $a = 1$, then $q(q-1) + (k-q)(k-q-a) \geqq k(k-a-1)/2$. If $a = 0$ equality holds if and only if $q = [(k+1)/2]$.*

If $a=1$ equality holds if and if $q=k/2$.

LEMMA 3.3. *If q, k are integers then $(q-1)^2+(k-q+1)(k-q-a) \geqq k(k-a-1)/2)$ for $a=0$ and all q, and for $a=1$ and all $q \neq (k+1)/2$. If $a=0$ equality holds if and only if $q=[(k+2)/2]$. If $a=1$ equality holds if and only if $q=k/2$ or $q=(k/2)+1$. For $a=1$, $q=(k+1)/2$, $(q-1)^2+(k-q+1)(k-q-1)=\{k(k-2)-1\}/2$.*

4. **Dimension and characterization of minimal completely separating bases and matrices.** We shall complete the proof of theorem 2.8 by induction, in this section. Before proceeding, however, we must first make use of the induction hypothesis in order to generalize Lemma 2.5 to matrices having n columns, where n is not necessarily a power of 2.

LEMMA 4.1. *Let s be a fixed positive integer. If it is true that $B_i^{(s+1)\times 2^s}$, and those matrices obtained by permuting the rows and columns of $B_i^{(s+1)\times 2^s}$, $i=1$ or 2, are the only minimal completely separating incidence bases for a matrix having 2^s columns, and if $B^{q\times n}$ is a minimal c.s.i.b. for n, $2^s<n\leqq 2^{s+1}$, then $q=s+2$.*

Proof. Obviously $q\leqq s+2$. Indeed by Lemma 2.4, $B_i^{(s+1)\times 2^s}$ is a c.s.i.b.; hence any restriction of $B_i^{(s+2)\times 2s+1}$ to n columns is a c.s.i.b., $i=1$ or 2.

It suffices to prove $q>s+1$ for $n^s=2^s+1$, since were there a c.s.i.b. $B^{(s+1)\times(2^s+k)}$ for any $k>1$, a restriction of this matrix to 2^s+1 columns would be completely separating for $k=1$.

Suppose $B\equiv B^{q\times(2^s+1)}$ is a c.s.i.b. and $q\leqq s+1$. Let $b_i\in B$, $b_i\neq(1,1,\cdots,1)$. Then $b_i^{j_1}=1$, $b_i^{j_2}=0$ for some j_1, j_2. Let B^* be the matrix obtained from B by deleting *column* j_1. Then B^* has q rows and 2^s columns, where $q\leqq s+1$; hence it is a minimal c.s.i.b. Therefore if the only minimal c.s.i.b. for $n=2^s$ are permutations of $B_i^{(s+1)\times 2^s}$, then B^* must be such a permutation. Therefore, any row of B^* not identically one has exactly 2^{s-1} ones and 2^{s-1} zeroes. Thus the ith row of B has $2^{s-1}+1$ ones and 2^{s-1} zeroes (since $b_i^{j_1}=1$). Similarly, if we delete column j_2, we find that the ith row of B has 2^{s-1} ones and $2^{s-1}+1$ zeroes (since $b_i^{j_2}=0$), and we arrive at a contradiction, thereby proving that $q=s+2$.

LEMMA 4.2. *Let $B^{q\times n}$ by a c.s.i.b. Then if $n>2^s$, $q\geqq s+2$. The proof is* trivial and will be omitted.

We shall now complete the proof of theorem 2.8 by induction on s. For $s=1$, the only possible row vectors of a basis are (1, 1), (0, 1), and (1, 0), and any two such vectors form a minimal basis of the form $B_i^{2\times 2}$, $i=1,2$.

Suppose the theorem is true for matrices having 2^t columns, $t<s$. We shall prove the theorem for matrices having 2^s columns, dividing the proof into five parts;

PART A. *Any* c.s.i.b. $B^{(s+1)\times 2^s}\equiv B$ satisfies both the 0-condition and the 2 condition. (See Definitions 1.4 and 1.5).

Proof of Part A. Suppose B does not satisfy the 0-condition. Then, without loss of generality we may assume that there exists a row b_i in B such that $b_i^j = 0$ for all $j \leqq 2^{s-1} + k$, $0 < k \leqq 2^{s-1}$, and $b_i^j = 1$ for all $j > 2^{s-1} + k$. B is a basis of a c.s.i.m. M.

Let B^* be the matrix obtained by restricting B to the first $2^{s-1} + k$ coluumns, with the ith row deleted.

Let M^* be the matrix obtained by restricting M to the first $2^{s-1} + k$ columns. Then B^* is a c.s.i.b. of M^* having the dimensions $s \times (2^{s-1} + k)$ which, by the induction hypothesis and Lemma 4.2, is impossible.

Suppose B does not satisfy the 2-condition. Then, without loss of generality we may assume that there exists 2 row vectors b_1, b_2 such that $b_1^j = b_2^j$ for $j = 1, \cdots, 2^{s-1} + p$, $p > 0$.

Let B^* be obtained by restricting B to the first $2^{s-1} + p$ columns, with the first row deleted.

Let M^* be obtained by restrincting M to the first $2^{s-1} + p$ columns. Then B^* is a c.s.i.b. (of M^*) of dimension $(s) \times (2^{s-1} + p)$, which, by the induction hypothesis and Lemma 4.2, is impossible.

Therefore B satisfies both the 0-condition and the 2-condition.

PART B. If $B^{(s+1)\times 2^s} = B$ is a c.s.i.b. and if there is a $b_i \in B$ such that b_i has exactly 2^{s-1} components equal to zero, $B \sim B_i^{(s+1)\times 2^s}$, $i = 1, 2$.

Proof of part B. Without loss of generality we may assume that $b_1 \in B$ and $b_1^j = \begin{Bmatrix} 0 & j \leqq 2^{s-1} \\ 1 & j > 2^{s-1} \end{Bmatrix}$ Let B^* be obtainedby restricting B to the first 2^{s-1} columns with the first row deleted.

Let M^* be obtained by restricting M to the first 2^{s-1} columns.

Then B^* is a c.s.i.b. of dimension $s \times 2^{-1}$, and therefore by the induction hypothesis, $B^* \sim B_i^{s\times 2^{s-1}}$, $i = 1$ or 2. Therefore there exists a column vector in B^* which is everywhere zero except in one component which component is equal to one. Therefore there is also a column vector in B which is everywhere zero except in one component, which component is equal to one.

Thus by Lemmas 2.9 and 2.11, $B \sim B_i^{(s+1)\times 2^s}$, i = 1 or 2.

PART C. Let $B \equiv B^{(s+1)\times 2^s}$ be a basis of a c.s.i. m. M. If there exists a row vector $b_{i_0} \in B$ such that $2^s > \sum_j b_i^j > 2^{s-1}$, then $2^s > \sum_j b_i^j > 2^{s-1}$ for *all* $i = 1, \cdots, s+1$, and if $m_k = (m_k^1, \cdots, m_k^{2^s}) \in M$ then $2^s > \sum_j m_k^j > 2^{s-1}$.

Proof of Part C. Suppose there exists a $b_{i_0} \in B$ such that $2^s > \sum_j b_{i_0}^j > 2^{s-1}$. $\sum_j b^j \geqq 2^{s-1}$ for every i, by part A.

If $\sum_j b_i^j = 2^{s-1}$ or 2^s for some i, then by Part B and Lamma 2.9 we know that $B \sim B_i^{(s+1)\times 2^s}$, $i = 1$ or 2 and then for each i, including i_0, $\sum_j b_i^j = 2^{s-1}$ or 2, contrary to our assumption. Therefore $2^s > \sum_j b_i^j > 2^{s-1}$ for every i. Let $m_k = \sum_i c_i b_i = (m_k^1, \cdots, m_k^{2s}) \in M$, $m_k \neq b_{i_0}$. Then there exists *a* j_0, $j_0 \neq i_0$, such that $c_{j_0} \neq 0$, and $\{b_1, \cdots, b_{j_0-1}, m_k, b_{j_0+1}, \cdots, b_{s+1}\}$ is also a c.s.i.b. containing b_{i_0}. Therefore by the first half of this proof, $2^s > \sum_j m_k^j > 2^{s-1}$.

PART D. If $B \equiv B^{(s+1)\times 2^s}$ is a c.s.i.b. of a c.s.i.m. M and if there exists a $b_i \in B$ such that $2^s > \sum_j b_i^j > 2^{s-1}$ then there is an $m_k \in M$, such that $m_k = \sum_i c_i b_i$ where $c_1 = -1$ and $\sum_i c_i = 0, 1$ or 2.

Proof of Part D. $(1,0,\cdots,0)' \notin B$ and $(0,0,\cdots,0)' \notin B$, Lemmas 2.2, 2.9 and 2.11. Let B^* be obtained by deleting the first row of B. $B^* \not\supset A^s$ since $(0,\cdots,0)' \notin B^*$, and therefore B^* has two identical columns $(b^{j_1})^* = (b^{j_2})^*$. Since by Lemma 2.2 B does not have two identical columns, we may assume that $b_1^{j_1} = 1$, $b_1^{j_2} = 0$. By the definition of a completely separating matrix, there must exist a row $m_k \in M$ where $m_k = \sum c_i b_i$ with $m_k^{j_1} = 0$, and $m_k^{j_2} = 1$. Consequently $m_k^{j_1} - m_k^{j_2} = -1 = \sum c_i (b_i^{j_1} - b_i^{j_2}) = c_1$.

By Lemmas 2.10 and 2.2, $(1,\cdots,1)' \notin B$ and $(0,\cdots,0)' \notin B$.
Therefore, there must be some pair of columns b^p and b^q, both in B, whose sum is the unit vector, for if not we would have in B, exactly one column of each of the 2^s such pairs. But we have exhibited one such pair, namely $(1,\cdots,1)'$ and $(0,\cdots,0)'$, neither of which is in B. It follows that $\sum_i c_i = 0, 1$ or 2 since $m_k = 0$ or 1 for every j, and $m_k^p + m_k^q = \sum c_i (b_i^p + b_i^q) = \sum_i c_i$.

PART E. If $B^{(s+1)\times 2^s} \equiv B$ is a basis of a c.s.i.m. M then $\sum_j b_i^j = 2^{s-1}$ or 2^s for any $b_i \in B$ and $B \sim B_i^{(s+1)\times 2^s}$, $i = 1$ or 2.

Proof of Part E. (by contradiction). Suppose there exists a $b_i \in B$ such that $\sum_j b_i^j \neq 2^{s-1}$, and $\sum_j b_i^j \neq 2^s$. We know $\sum_j b_i^j \geqq 2^{s-1}$ (Part A, by the 0-condition) Therefore $2^s > \sum_j b_i^j > 2^{s-1}$. Then there exists an $m_v = \sum_j c_j b_j = (m_v^1, \cdots, m_v^{2^s}) \in M$ such that $2^s > \sum_j m_v^j > 2^{s-1}$ and $c_1 = -1$, $\sum_i c_i = 0, 1$ or 2 (by Parts C and D). Thus there must be *more* than 2^{s-1} columns b^p in B for which $\sum_i c_i b_i^p = m_v^p = 1$. We shall prove that there are *at most* 2^{s-1} such columns. Before proceeding, we shall first give an example.

EXAMPLE. Let $\sum_i c_i = 1$, and let there be only one c_i which is equal to -1. Without loss of generality we may assume that $c_1 = -1$, $c_2 = c_3 = 1$ and $c_i = 0$ for $i > 3$. Let b^p be a column in B. Then $\sum_i c_i b_i^p = 0$ or 1. If $\sum_i c_i b_i^p = 1$, the first three components of b^p must be as in one of the following three cases: (1) $b^p = (1\ 1\ 1\ \cdots)'$, (2) $b^p = (0\ 1\ 0\ \cdots)'$, or (3) $b^p = (0\ 0\ 1\ \cdots)'$ and if $\sum_i c_i b_i^p = 0$, the first three components must be as in: (4) $b^p = (0\ 0\ 0\ \cdots)'$, (5) $b^p = (1\ 1\ 0\ \cdots)'$ or (6) $b^p = (1\ 0\ 1\ \cdots)'$. We shall say that a column is of *type* i if and only if its first three components are as in case (i). Suppose B has t_i columns of type i, $i = 1, \cdots 6$.

(1.1) $t_i \leqq 2^{s-2}$ for all i; since each column type has three fixed components, and the remaining $s-2$ components may be either 0 or 1.

(1.2) $\Sigma_i t_i = 2^s$. The matrix B has the following form:

$$
B = \begin{array}{c} b_i \\ b_2 \\ b_3 \\ \\ \\ \\ b_{s+1} \end{array}
\begin{bmatrix} 1\cdots 0\cdots 0\cdots 0\cdots 1\cdots 1\cdots \\ 1\cdots 1\cdots 0\cdots 0\cdots 1\cdots 0\cdots \\ 1\cdots 0\cdots 1\cdots 0\cdots 0\cdots 1\cdots \\ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot \\ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot \\ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot \\ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot \end{bmatrix}
\begin{array}{l} c_1 = -1 \\ c_2 = 1 \\ c_3 = 1 \\ \\ \\ \\ c_i = 0 \end{array}
$$

$$\sum_i c_i b_i = m_v = \quad (1\cdots 1\cdots 1\cdots 0\cdots 0\cdots 0\cdots\ )$$

where each column type (i) appears t_i times. Thus the nunmber of components in rows b_1 and b_2 which are identical is $t_1 + t_3 + t_4 + t_5$; the number of components in rows b_1 and b_3 which are identical is $t_1 + t_2 + t_4 + t_6$. By Part A, the 2-condition, we find that $2t_1 + t_2 + t_3 + 2t_4 + t_5 + t_6 \leqq 2^s$. Together with equation (1.2) this gives $t_1 = t_4 = 0$. Using inequality (1.1) we find that there are at most $t_2 + t_3 \leqq 2^{s-1}$ columns b^p in B such that $\Sigma_i c_i b_i^p = 1$, contradictory to our assumption that $\Sigma_j m_v^j > 2^{s-1}$.

We now proceed to the proof of part E. We may assume, without loss of generality, that the rows of B are so ordered that $c_i = -1$ for $1 \leqq i \leqq h$, $c_i = 1$ for $h < i \leqq h + k$ and $c_i = 0$ for $i > h + k$. Let $I_j = \{i : c_i = j\}$, $I^p = \{i : b_i^p = 1\}$. Let $|A|$ denote the number of elements in the set A. By Lemma 2.6 and part D of this theorem, $|I_{-1}| = h \neq 0$, $|I_1| = k \geqq h$ and $|I_j| = 0$ if $j \neq 0, 1, -1$.

Let X be the set of all possible column vectors α^j of length $s + 1$, satisfying (a) $\alpha_i^j = 0$ or 1 for $1 \leqq i \leqq s + 1$, and (b) $\Sigma_i \alpha_i = 0$ or 1. The columns of B form a proper subset of X. We shall define a partition of X into equivalence classes called *types*; two columns α^p, α^t in X belong to the same type if and only if $\alpha_i^p = \alpha_i^t$ for all $i \in I_1 U I_{-1}$. This is obviously an equivalence relation. Let $A = \{\alpha\}$ be the family of types. We shall now partition A into two equivalence classes A_0 and A_1, where $A_i = \{\alpha : \alpha \in A$ and if α^p is a representative of type α, then $\Sigma_j c_j \alpha_j^p = i\}$. This definition is independent of the choice of a representative, for if α^p and α^t are both of type α, then $\Sigma_j c_j \alpha_j = \Sigma_j c_j \alpha_j$. $A_0 \cap A_1 = \phi$, and $A_0 \cup A_1 = A$, since by definition $\Sigma_j c_j \alpha_j^p = 0$ or 1.

If $\alpha^p \in \alpha \in A_0$ then there exists an integer q, $1 \leqq q \leqq h + 1$, such that (1.3) $|I^p \cap I_{-1}| = |I^p \cap I_1| = q - 1$ and if $\alpha^p \in \alpha \in A_1$, there exists an integer q, $1 \leqq q \leqq \mathrm{Min}(h + 1, k)$, such that (1.4) $|I^p \cap I_{-1}| = q - 1$ and $|I^p \cap I_1| = q$. We shall use these q's to partition the sets A_0 and A_1. Let $A(q, i) = \{\alpha : \alpha \in A_i$ and if α^p is a representative of type α, then $|I^p \cap I_{-1}| = q - 1\}$. This definition is independent of the choice of representative, for if α^p and α^tboth belonge to the class α, $I^p = I^t$. The sets $A(q,i)$, $1 \leqq q \leqq \min(h+1,k)$ are disjoint and their union is A_i, $i=0$, 1. (1.5) $|A(q,i)| = \binom{h}{q-1}\binom{k}{q-1+i}$. The number of columns $\alpha^i \in X$ which are of type α is (1.6) $|\alpha| = 2^{s+1-(h+k)}$. We shall denote by B_1 the set of columns $b^p \in B$ for wnich $\Sigma c_i b_i^p = m_v^p = 1$. Obviously $B_1 = \{\alpha^p : \alpha^p \text{ is a representative of } \alpha, \alpha \in A(q,1) \text{ for some } q, 1 \leqq q \leqq \min(h+1,K)\}$, and consequently (1.7) $|B_1| \leqq |\alpha| \; \Sigma_q |A(q,1)|$.

If a columns α^p is a representative of the class α we shall say that α^p *is of type* α, and if two columns belong to the same class α we shall say that they are *of the same type*.

Let t_α be the number of columns of type α which are in the matrix B. Consider any pair of row vectors b_j, b_w in B such that $j \in I_{-1}$, $w \in I_1$. If $b_j^p = b_w^p$ for *some* column $b^p \in B$, then $b_j^u = b_w^u$ for every column b^u in B of the same type as b^p. Thus we may define the set C_{wj} of all the types α for which the j^{th} and w^{th} components of any column of type α are equal. Thus $C_{wj} = \{\alpha : b_j^p = b_w^p \text{ if } b^p \text{ is of type} \alpha\}$. By part A, the 2-condition, we find that the number of identical components in rows b_j and b_w is

$$(1.8) \qquad \sum_{\alpha \in Cwj} t_\alpha \leqq 2^{s-1}.$$

Summing over all kh pairs (w,j), we arrive at

$$(1.9) \qquad \sum_{\substack{w \in I_1 \\ j \in I_{-1}}} \sum_{\alpha \in Cwj} t_\alpha = \sum_{\alpha \in A} a_\alpha t_\alpha \leqq k h 2^{s-1}$$

where a_α simply denotes the numerical coefficient of t_α in the expansion. The total number of columns in B is

$$(1.10) \qquad \sum_{\alpha \in A} t_\alpha = 2^s.$$

For any α, a_α is actually the number of pairs (w,j) for which $\alpha \in C_{wj}$. Let $\alpha \in A(q,i)$ and let b^p be a representative of α. Then $\alpha \in C_{wj}$ if and only if $b_w^p = b_j^p$. There are exactly $(q-1)(q-1+i)$ pairs (w,j) such that $b_w^p = b_j^p = 1$ and $(k-q+1-i)$ $(h-q+1)$ pairs such that $b_w^p = b_j^p = 0$. Thus, for any q.

$$(1.11) \qquad a_\alpha = q(q-1) + (k-q)(h-q+1) \quad \text{if } \alpha \in A(q,1)$$

$$(1.12) \qquad a_\alpha = (q-1)^2 + (k-q+1)(h-q+1) \quad \text{if } \alpha \in A(q,0).$$

We shall distinguish three cases:

Case (1). $\Sigma_i c_i = 0$. In this case $h = k$, $|\alpha| = 2^{s+1-2k}$and $|A(q,1)| = \binom{k}{q-1}\binom{k}{q}$. $\Sigma_q \binom{k}{q-1}\binom{k}{q} = \binom{2k}{k-1} \leqq 2^{2k-2}$ (Lemma 3.1) (See [1] p. 48). Therefore $|B_1| \leqq 2^{s-1}$ (Equation 1.7) contrary to the assumption that $\Sigma_p m_v^p > 2^{s-1}$.

Case (2). $\Sigma_i c_i = 1$. Here $h = k - 1$ and equations (1.11) and (1.12) become: $a_\alpha = q(q-1) + (k-q)^2$ if $\alpha \in A(q,1)$ and $a_\alpha = (q-1)^2 + (k-q+1)(k-q)$ if $\alpha \in A(q,0)$. By Lemmas 3.2 − 3, $a_\alpha \geqq \dfrac{k(k-1)}{2}$ for all $\alpha \in A$, while for $\alpha \in A(q,1)$, $a_\alpha = \dfrac{k(k-1)}{2}$ if and only if $q = \left[\dfrac{k+1}{2}\right]$. Using equation (1.10) we find that $\Sigma_{\alpha \in A} a_\alpha t_\alpha \geqq k(k-1)2^{s-1}$ which, together with (1.9) means that $\Sigma_{\alpha \in A} a_\alpha t_\alpha = = k(k-1)2^{s-1}$. Therefore $t_\alpha \neq 0 \Rightarrow a_\alpha = \dfrac{k(k-1)}{2}$, and thus if $\alpha \in A(q,1)$, $q = \left[\dfrac{k+1}{2}\right]$. Consequently if $b^p \in B_1$, b^p must be of type α, $\alpha \in A(q,1)$. If *k is even*, then $q = \dfrac{k}{2}$ and $\left| A\left(\dfrac{k}{2}, 1\right) \right| = \begin{pmatrix} k-1 \\ \frac{k}{2}-1 \end{pmatrix} \begin{pmatrix} k \\ \frac{k}{2} \end{pmatrix} \leqq 2^{2k-3}$ (Lemma 3.1), and $|\alpha| = 2^{s+1-(2k-1)} = 2^{s+2-2k}$. Therefore $B_1 \leqq 2^{s-1}$ contrary to the assumption that $\Sigma_p m_v^p > 2^{s-1}$. If *k is odd*, $q = \dfrac{(k+1)}{2}$ and $\left| A\left(\dfrac{k+1}{2}, 1\right)\right| = \begin{pmatrix} k-1 \\ \frac{k-1}{2} \end{pmatrix} \begin{pmatrix} k \\ \frac{k+1}{2} \end{pmatrix} \leqq 2^{2k-3}$ (Lemma 3.1), $|\alpha| = 2^{s+2-2k}$ and $|B_1| \leqq 2^{s-1}$ which, as above, is a contradiction.

Case (3). $\Sigma_i c_i = 2$. In this case $h = k-2$ and equations (1.11) and (1.12) become $a_\alpha = q(q-1) + (k-q-1)(k-q)$ if $\alpha \in A(q,1)$ and $a_\alpha = (q-1)^2 + + (k-q-1)(k-q+1)$ if $\alpha \in A(q,0)$. For *k even*, $a_\alpha \geqq \dfrac{k(k-2)}{2}$ and if $\alpha \in A(q,1)$ equality holds if and only if $q = \dfrac{k}{2}$ (Lemmas 3.2–3.) Thus $t_\alpha \neq 0$ and $\alpha \in A(q,1) \Rightarrow a_\alpha = \dfrac{k(k-2)}{2} \Rightarrow \alpha \in A\left(\dfrac{k}{2}, 1\right)$. $\left|A\left(\dfrac{k}{2},1\right)\right| = \begin{pmatrix} k-2 \\ \frac{k}{2}-1 \end{pmatrix} \begin{pmatrix} k \\ \frac{k}{2} \end{pmatrix} \leqq$ $\leqq 2^{2k-4}$ (Lemma 3.1), and $|\alpha| = 2^{s+1-(2k-2)} = 2^{s-2k+3}$. Therefore $|B_1| \leqq 2^{s-1}$. If *k is odd* $a_\alpha > \dfrac{k(k-2)}{2}$ for $\alpha \in A(q,1)$. Therefore $a_\alpha - 1 \geqq \dfrac{k(k-2)}{2} - \dfrac{1}{2}$ for $\alpha \in A(q,1)$. Also $a_\alpha \geqq \dfrac{k(k-2)-1}{2}$ if $\alpha \in A(q,0)$, by Lemmas 3.2–3. Consequently $\Sigma_{\alpha \in A_1} [a_\alpha - 1]\, t_\alpha + \Sigma_{\alpha \in A_0} a_\alpha t_\alpha \geqq \dfrac{k(k-2)-1}{2} \cdot 2^s$. But $k(k-2)2^{-1} = = \dfrac{[k(k-2)-1]}{2} \cdot 2^s + 2^{s-1}$, and by equation (1.9) $\Sigma_{\alpha \in A} a_\alpha t_\alpha \leqq k(k-2)2^{s-1}$. Therefore $\Sigma_{\alpha \in A_1} t_\alpha \leqq 2^{s-1}$, and $|B_1| \leqq 2^{s-1}$, which contradicts the assumption that $\Sigma_p m_v^p > 2^{-1}$.

We have proved in each case that the assumption that there exists a $b_i \in B$ such that $\sum_j b_i^j \neq 2^{s-1}$ and $\sum_j b_i^j \neq 2^s$ leads to a contradiction, completing the proof of part E.

To conclude the proof of the theorem we note that since there is only one row vector b_i, namely the unit vector, for which $\sum_j b_i^j = 2^s$, there must be a vector b_i in B such that $\sum_j b_i^j = 2^{s-1}$. We then apply part B to complete the proof by induction.

Theorem 2.8 and Lemma 4.1 may be combined into:

THEOREM 4.3. *If* $B^{q \times n} = B$ *is a minimal c.s.i.b. and* $2^{s-1} < n \leqq 2^s$, *then* $q = s + 1$. *If* $n = 2^s$ *then* $B \sim B_i^{(s+1) \times 2^s}$ *for* $i = 1$ *or* 2.

THEOREM 4.4. *There is only one c.s.i.m. M (up to row and column permutations) having* 2^s *columns which is minimal in the sense that both its basis is minimal and that no subset of rows of M is completely separating. M is the matrix whose first s rows* $m_1, \cdots m_s$ *are identical to the first s rows of* $B_1^{(s+1 \times 2^s)}$ *and whose remaining s rows are* $m_{s+i} = (1, \cdots 1) - m_i$, $i = 1, \cdots s$.

Proof. Let N be any c.s.i.m. having 2^s columns, whose basis is minimal. Then $B_1^{(s+1 \times 2^s)} = B$ is a basis of N, for $B_2^{(s+1 \times 2^s)}$ is the only other minimal basis, and if $B_2^{(s+1 \times 2^s)}$ is a basis of N, then so is $B_1^{s+1 \times 2^s)}$. Let b_i, $i = 1, \cdots s + 1$, be the rows of B as defined in (1.8). We shall first prove that for any row n_k of N, $n_k = b_i$ or $n_k = b_{s+1} - b_i$ for some i, $i = 1, \cdots s$, where b_{s+1} is the unit vector. $n_k = \sum_i c_i b_i$ and since $(0, \cdots 0, 1)' \in B$ and $c_i = 0$, 1 or -1 by Lemma 2.6, we have $c_{s+1} = 0$ or 1. Suppose $c_{s+1} = 1$ and suppose $c_i = 1$ for some $i \neq s + 1$. Then there is some column b^j in B which is everywhere 0 except in the ith and $s + 1$st components, giving $n_k^j = 2$, which is impossible. Similarly there is at most one c_i which is equal to -1, for if there wre more than one such c_i we should get $n_k^j = -1$ for some j, which is impossible. Therefore if $c_{s+1} = 1$, then $n_k = b_{s+1}$, or $n_k = b_{s+1} - b_i$ for some $i \leqq s$. Similarly if $c_{s+1} = 0$, then no c_i can be equal to -1 and at most one c_i can be equal to 1. Thus $n_k = b_i$ for some $i \leqq s$. We have proved that the rows of N form a subset of the rows of M with the unit vector added. we shall now show that if N is a proper subset of M, it is not completely separating. Suppose N is a proper subset of M. By symmetry, we may assume that m_1 is not in N. Then if $n_k = b_i$, $n_k^1 = 1 = n_k^{2^{s-1}+1} = 1$ and if $n_k = b_{s+1} - b_i$, then $n_k^1 = 0$. Therefore there does not exist an n_k in N such that $n_k^1 = 1$ and $n_k^{2^{s-1}+1} = 0$, which means that N is not a completely separating matrix. In Lemma 2.4 we proved that M with the unit vector added is completely separating. Obviously the unit vector is superfluous, and we have proved that if N is any c.s.i.m. whose basis is minimal and which itself is minimal in the sense that no subset of its rows is completely separating, then $N \sim M$.

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